Increasing and Decreasing Functions

Definition. A function is increasing where $f'(x) > 0$ and decreasing where $f'(x) < 0$.

Example

Find the values of x for which $y = x^3 - 6x^2 - 15x$ is decreasing.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 12x - 15 = 3(x-5)(x+1) < 0 \implies -1 < x < 5$$

Example

$f(x) = x^5 - \dfrac{3}{x}$, $x \neq 0$. Show that f is increasing for all x in its domain.

$f'(x) = 5x^4 + \dfrac{3}{x^2}$. *Both terms are positive for all $x \neq 0$, so $f'(x) > 0$.*

Example

Find the range of values of k for which $f(x) = x^3 + kx^2 + 3x$ is increasing for all real x.

$f'(x) = 3x^2 + 2kx + 3 \geq 0$ *for all x: discriminant ≤ 0.*
$4k^2 - 36 \leq 0 \implies -3 \leq k \leq 3$

Textbook Exercises: SPS Course 6.1, Exercise 4A

Stationary Points

Definition. A stationary point of $y = f(x)$ is a point where $f'(x) = 0$: a maximum, a minimum, or a stationary point of inflection.

Fact (Classifying with the second derivative) — At a stationary point $x = a$:

$f''(a) > 0$	minimum
$f''(a) < 0$	maximum
$f''(a) = 0$	no conclusion — check the gradient either side

Knowledge of the shape of the graph (e.g. a positive cubic) is an acceptable alternative justification.

Example

Find the stationary points of $y = x^3 - 6x^2 + 9x + 2$ and determine their nature. Hence sketch the curve.

$\frac{dy}{dx} = 3x^2 - 12x + 9 = 3(x-1)(x-3) = 0 \implies x = 1, 3$

$\frac{d^2y}{dx^2} = 6x - 12$: *at* $x = 1$, $-6 < 0$: *maximum* $(1, 6)$; *at* $x = 3$, $6 > 0$: *minimum* $(3, 2)$.

Sketch: positive cubic through $(0, 2)$, *rising to* $(1, 6)$, *falling to* $(3, 2)$, *rising.*

Example (Edexcel C2)

The curve with equation $y = x^2 - 32\sqrt{x} + 20,\ \ x > 0$, has a stationary point P. Use calculus to find the coordinates of P, and determine its nature.

$\frac{dy}{dx} = 2x - 16x^{-1/2} = 0 \implies 2x = \frac{16}{\sqrt{x}} \implies x^{3/2} = 8 \implies x = 4$

$y = 16 - 64 + 20 = -28$: $P(4, -28)$.

$\frac{d^2y}{dx^2} = 2 + 8x^{-3/2} > 0$: *minimum.*

Textbook Exercises: SPS Course 6.1, Exercise 4B and Exercise 5 Q1–6

Using Stationary Points

Example

The curve $y = 2x^3 + ax^2 + bx$ has a stationary point at $(1, -4)$.

1. Find a and b.
2. Find the other stationary point and determine the nature of both.

1. *On the curve:* $2 + a + b = -4$. *Stationary there:* $\frac{dy}{dx} = 6x^2 + 2ax + b$ *gives* $6 + 2a + b = 0$. *Subtracting:* $4 + a = 4 \implies a = 0$, *then* $b = -6$.
2. $y = 2x^3 - 6x$: $\frac{dy}{dx} = 6x^2 - 6 = 0 \implies x = \pm 1$. *Other point* $(-1, 4)$. $\frac{d^2y}{dx^2} = 12x$: $(1, -4)$ *minimum,* $(-1, 4)$ *maximum.*

Example

Show that the curve $y = x^4 - 4x^3 + 16x - 3$ has a stationary point at $x = -1$, and determine its nature.

$\frac{dy}{dx} = 4x^3 - 12x^2 + 16$; *at* $x = -1$: $-4 - 12 + 16 = 0$.
$\frac{d^2y}{dx^2} = 12x^2 - 24x$; *at* $x = -1$: $12 + 24 = 36 > 0$: *minimum.*

Exercise. $y = x^3$ has $\frac{dy}{dx} = 0$ at the origin, yet the origin is neither a maximum nor a minimum. Check the gradient on each side of $x = 0$, sketch the curve, and explain why the second-derivative test fails here.

Textbook Exercises: SPS Course 6.1, Exercise 5 (remaining) and Exercise 6